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The Oort cloud as a dynamical system

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Abstract

The Oort cloud is a formally simple dynamical system if we restrict its study to the heliocentric Kepler problem perturbed by the quadratic tidal potential of the Galaxy. Nonintegrability of this system leads to rich dynamics, shaped by the resonances of various nature. A new perturbation solution has helped to identify a class of resonances between the precessing nodes and the rotation of the Galaxy.

1 Introduction

In 1950, J. H. Oort [10] proposed a hypothesis that long-periodic comets observed in the Solar System come from a cloud surrounding the Sun, instead of being some captured interstellar objects. This bold hypothesis, although initially backed up by the observations of merely 19 objects, has nowadays gained a common acceptance: the cloud is assumed to be a shell ranging from 50 to 100 kAU, populated by 10^{12} objects. What changed since the times of Oort, is the identification of the mechanism responsible for the transport of comets to the inner Solar System. Curiously, Oort – an expert in galactic dynamics – thought about the sporadic encounters with passing-by stars, whereas today the Galactic potential is considered the primary factor. The influence of our Galaxy is the only significant factor of systematic nature that can influence cometary orbits. Other phenomena, like encounters with other stars and molecular clouds are by no means negligible, but they happen occasionally as a quasi-random forcing superimposed on the steady Galactic force background. For these reasons, a good understanding of a dynamical system resulting from the combination of heliocentric Keplerian motion with the Galactic potential perturbations is of primary importance to our understanding of the Solar System.

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There are many elaborated models describing the gravity field of our Galaxy. But any such model, however sophisticated, enters the equations of motion for a comet only through a difference between the forces exerted on the Sun and on the comet. In this context, a tidal approximation, leading to a quadratic potential, is a reasonable and satisfactory approximation. Assuming a reference frame that rotates with the Galaxy, we can even get rid of the explicit time dependence and the resulting Hamiltonian function is simply $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, where $\mathcal{H}_0$ is the usual Keplerian function

$$\mathcal{H}_0 = \frac{1}{2}\left(X^2 + Y^2 + Z^2\right) - \frac{\mu}{\sqrt{x^2 + y^2 + z^2}}, \tag{1}$$

in terms of the heliocentric Cartesian coordinates x, y, z and their conjugate momenta X, Y, and Z. If we use the units of Solar mass, $10^6\,$y, and $10^3\,$AU, the heliocentric gravitational constant $\mu \approx 4 \times 10^4\,\mathrm{M}_\odot^{-1}\,\mathrm{kAU}^3\,\mathrm{My}^{-2}$.

The treatment of the Galactic tide $\mathcal{H}_1$ depends on the assumed approximation level.

1. In principle, the complete Galactic tide is given in a rotating reference frame with the axis Oz normal to the Galactic disk plane. The potential is

$$\mathcal{H}_1 = -\Omega_0\,(xY - yX) + \frac{1}{2}\,\left(G_1 x^2 + G_2 y^2 + G_3\,z^2\right), \tag{2}$$

   where the constant G_i are time-dependent – mostly due to the variations of local stellar density resulting from the inclination of the solar orbit with respect to the Galactic disk. If the Oz axis points towards the northern Galactic pole, the reference frame rotation rate Ω_0 is negative.

2. Using few assumptions justified by our knowledge of G_i, the tidal potential (2) is usually reduced to

$$\mathcal{H}_1 = -\Omega_0\,(xY - yX) + \frac{1}{2}\,\left(G_2(y^2 - x^2) + G_3\,z^2\right), \tag{3}$$

   with constant values $G_2 \approx 7.1 \times 10^{-4}\,\mathrm{My}^{-2}$, $G_3 \approx 5.7 \times 10^{-3}\,\mathrm{My}^{-2}$, and additionally,

$$\Omega_0 \approx -\sqrt{G_2} \approx -2.7 \times 10^{-2}\,\mathrm{My}^{-2}. \tag{4}$$

3. Observing that G_3 is almost ten times bigger than G_2, the perturbing Hamiltonian $\mathcal{H}_1$ is often reduced to the simple form

$$\mathcal{H}_1 = \frac{1}{2}\,G_3\,z^2. \tag{5}$$

   This model is called the Galactic disk tide and it can be used in a reference frame with fixed direction of axes.

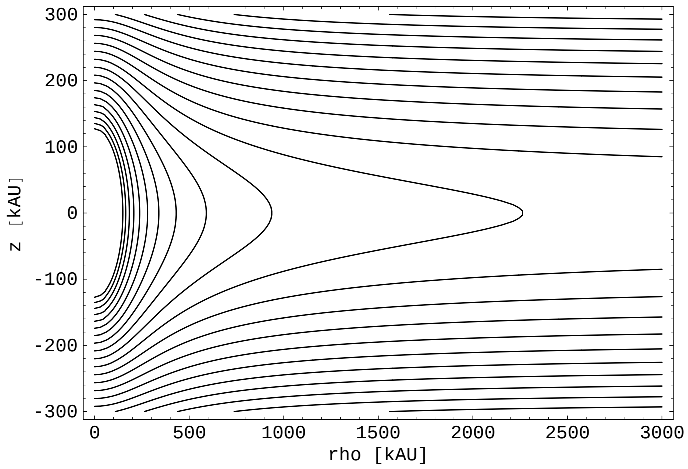

Figure 1.— Zero velocity curves for the Galactic disk tides problem on the plane of $\rho = \sqrt{x^2+y^2}$ and z.

The present paper does not aim at a complete review of the Oort cloud dynamics; many (if not most of) fundamental papers will remain unquoted. The brief synthesis aims at providing the basic facts that concern the motion in the gravity field of the Sun and the Galactic tide seen as a dynamical system. Using a probably controversial statement, the paper aims at liberating the Oort cloud from comets in order to make it fancier looking for theoretically oriented *aficionados* of dynamics. Occasional remarks about what has not been done yet are intended to encourage interested readers.

2 Galactic disk tides

At the first glimpse, the galactic disk tide looks like a special case of the generalized van der Waals problem, sharing its symmetry with respect to the rotations around the Oz axis. But the direct reduction of $\mathcal{H}_1 = \frac{1}{2} G_3 z^2$ to the potential

$$\mathcal{H}_{\mathrm{vdW}} = \alpha(x^2+y^2+\beta z^2)/2,$$

is not possible, because one cannot choose α and β in a way that disables the x^2+y^2 but leaves a nonzero z^2 contribution. Similarly to the generalized van der Waals problem, Galactic disk tides are nonintegrable. Due to the axial symmetry leading to the integral of motion

$$H = xY - yX = \sqrt{\mu\, a(1-e^2)}\, \cos I = \mathrm{const}, \tag{6}$$

the problem has effectively two degrees of freedom. The strict nonintegrability proof has not been given yet, but the Poincaré sections presented by Maciejewski and Pretka [8] do not leave any hope for the existence of another integral of motion apart from the Hamiltonian function. In these circumstances, there are only few strict results concerning the motion:

1. Two cases of the planar orbits exist: polar orbits (nonintegrable), and Keplerian orbits in the Galactic plane Oxy.

2. The zero velocity surfaces (Fig. 1) imply that the motion is bounded if the total energy is negative, i.e.

$$E = \frac{1}{2}\left(X^2+Y^2+Z^2\right) - \frac{\mu}{\sqrt{x^2+y^2+z^2}} + \frac{1}{2}\,G_3\,z^2 < 0. \tag{7}$$

3. The minimum heliocentric distance is reached when a comet crosses the plane $z=0$. There exists a lower bound on $r_{\min}$ for a given energy E and the third component of angular momentum H [8].

Surprisingly, the system has not attracted the attention of periodic orbits hunters.

Sufficiently close to the Sun, or – what is less often remembered – sufficiently close to the $z=0$ plane, one can use perturbation methods to gather more information about the problem. The Delaunay normalization removes one degree of freedom, producing a truncated Hamiltonian that is integrable. All known papers since Heisler and Tremaine [7] rely on the first order approximation, exploring the Hamiltonian

$$\mathcal{K}_1 = G_3\,\frac{L^2\,(G^2-H^2)}{4\,G^2\,\mu}\,\left(G^2 + 5\,(L^2-G^2)\,\sin^2 g\right), \tag{8}$$

in terms of the Delaunay variables. The level curves of this function reveal the possibility of libration and circulation of the argument of perihelion g, resembling to some extent the Lidov-Kozai resonance. Using the analogy with the treatment of the generalized van der Waals problem, Breiter, Dybczynski and Elipe [2] discussed the bifurcations of equilibria including the polar orbits, and offered a geometrical description of motion in terms of the Laplace vector $\boldsymbol{e}$ components. As it was realized later, the application of the Laplace vector makes miracles for the elegance and simplicity of the problem. Breiter and Ratajczak [4, 5] used the vectorial elements, involving the scaled angular momentum $\boldsymbol{h} = (\boldsymbol{r}\times\boldsymbol{R})/\sqrt{\mu a}$ and the Laplace vector

$$\boldsymbol{v} = (h_1, h_2, h_3, e_1, e_2, e_3)^{\mathrm{T}}. \tag{9}$$

They introduced a noncanical Poisson bracket

$$(f;g) \equiv \left(\frac{\partial f}{\partial \boldsymbol{v}}\right)^{\mathrm{T}} \mathbf{J}(\boldsymbol{v})\,\frac{\partial g}{\partial \boldsymbol{v}}, \tag{10}$$

with the structure matrix

$$\mathbf{J}(\boldsymbol{v}) = \begin{pmatrix} \hat{\mathbf{h}} & \hat{\mathbf{e}} \\ \hat{\mathbf{e}} & \hat{\mathbf{h}} \end{pmatrix}, \tag{11}$$

where the 'hat map' of any vector $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ is defined as

$$\hat{\mathbf{x}} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}. \tag{12}$$

This matrix is known as the vector product matrix, because

$$\hat{\mathbf{x}}\, \boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}. \tag{13}$$

Using the Lie-Poisson bracket (10), the equations of motion for the vectorial elements can be written in the non-canonical Hamiltonian form

$$\boldsymbol{v}' = (\boldsymbol{v};\, \mathcal{M}_1), \tag{14}$$

where derivatives with respect to $\tau = G_3 n^{-1} t$ are marked by the 'prime' symbol and the scaled Hamiltonian $\mathcal{M}_1$ is

$$\mathcal{M}_1 = -\frac{\mathcal{H}_1}{n\, a^2} = -\tfrac{1}{4}\left(h_1^2 + h_2^2 + 5\, e_3^2\right). \tag{15}$$

The resulting equations of motion are nonsingular and do not involve any transcendental functions, thus being a perfect formulation for Hamiltonian numerical integrators. The bracket (10) obeys the two constraints

$$\boldsymbol{e} \cdot \boldsymbol{h} = 0, \quad e^2 + h^2 = 1. \tag{16}$$

Thus only 4 out of 6 components of $\boldsymbol{v}$ are independent and the phase space is $\mathcal{S}^2 \times \mathcal{S}^2$. Using redundant variables is the usual price for nonsingularity.

Matese and Whitman [9] found the analytical solution for G, generated by Eq. (8), in terms of the Jacobian elliptic functions. According to their recommendation, g could be evaluated from the energy integral, although the latter only gives $\sin^2 g$ with an inevitable ambiguity in the value of the argument of perihelion. Their solution has been in common use for many years, but only recently Breiter and Ratajczak [4, 5] provided the two missing pieces: an unambiguous solution for g and the solution describing the longitude of the node Ω as the elliptic integral of the third kind. The latter has always been considered unimportant, because of the rotational symmetry, but knowing the motion of the nodes occurs fairly important in the understanding of the full tide problem.

3 Galactic tides

Probably the best analogy for the Galactic tides with the Hamiltonian $\mathcal{H}_1$ given by Eq. (3) is the three-dimensional Hill problem. The rotational symmetry of the Galactic

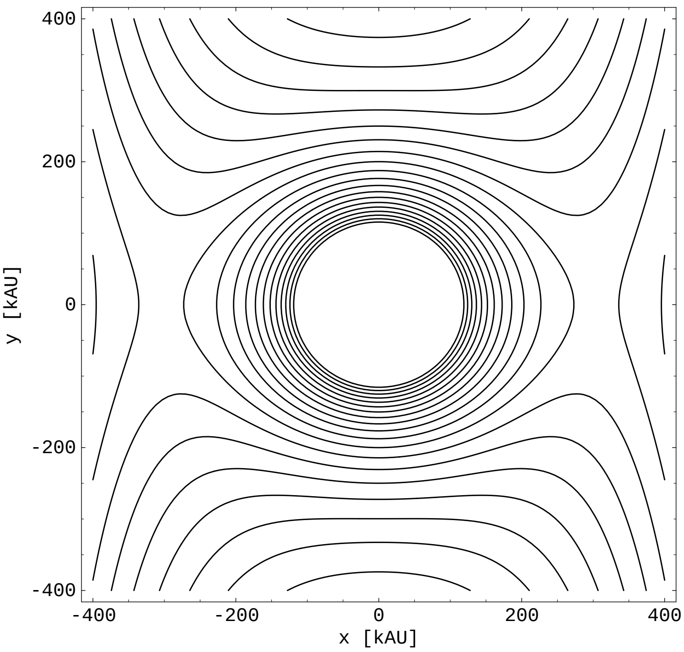

Figure 2.— Zero velocity curves for the Galactic tides problem on the plane $z = 0$.

disk tide problem is lost and even the Delaunay normalization brings us to the nonintegrable problem with two degrees of freedom. Thus the only strict results available in this case are based on the zero velocity surfaces defined by

$$\frac{\mu}{\sqrt{x^2+y^2+z^2}} + G_2 x^2 - \frac{1}{2} G_3 z^2 = C. \tag{17}$$

As seen in Fig. 2, there are natural bounds on the Oort cloud orbits, which cannot extend beyond 300 kAU [7] in the direction of the Galactic center.

Once more, no strict proof of nonintegrability has been given neither for the complete, nor for the Delauney normalized system, but the computation of Lyapunov exponents done by Brasser [1] leaves no hope for integrability in the unnormalized system. Recent, yet unpublished, results of Breiter, Fouchard, and Ratajczak show that positive Lyapunov exponents appear also in the first order normalized system. Similarly to the Galactic disc tide, the study of the normalized system can best be done in the vectorial elements (9), where the normalized Hamiltonian takes a simple form

$$\begin{aligned} \mathcal{M}_1 \;=\; & -\Big[\tfrac{5}{4}\,e_3^2 + \tfrac{1}{4}h_1^2 + \tfrac{1}{4}h_2^2 + \\ & + \nu\left(-\tfrac{5}{4}\,e_1^2 + \tfrac{5}{4}\,e_2^2 + \tfrac{1}{4}h_1^2 - \tfrac{1}{4}h_2^2 - n\,\Omega_0^{-1}\,h_3\right)\Big], \end{aligned} \tag{18}$$

with

$$\nu = \frac{\Omega_0^2}{G_3} = \frac{G_2}{G_3} \approx 0.125. \tag{19}$$

The resulting equations of motion, derived by Breiter et el. [3] are

$$h_1' = -\frac{5}{2}\,(1-\nu)\,e_2\,e_3 + \frac{1-\nu}{2}\,h_2\,h_3 + \frac{n\,\nu}{\Omega_0}\,h_2,$$

$$
\begin{aligned}
h_2' &= \frac{5}{2}(1+\nu)\, e_1\, e_3 - \frac{1+\nu}{2}\, h_1\, h_3 - \frac{n\,\nu}{\Omega_0}\, h_1, \\
h_3' &= \nu\,(h_1\, h_2 - 5\, e_1\, e_2), \\
e_1' &= -\frac{4+\nu}{2}\, h_2\, e_3 + \frac{5}{2}\,\nu\, h_3\, e_2 + \frac{n\,\nu}{\Omega_0}\, e_2, \\
e_2' &= \frac{4-\nu}{2}\, h_1\, e_3 + \frac{5}{2}\,\nu\, h_3\, e_1 - \frac{n\,\nu}{\Omega_0}\, e_1, \\
e_3' &= \frac{1-4\,\nu}{2}\, h_1\, e_2 - \frac{1+4\,\nu}{2}\, h_2\, e_1.
\end{aligned}
\tag{20}
$$

Setting $\nu = 0$ we obtain the equations of motion for the Galactic disk tide. The equations are nonsingular and an associated variational equations system is even simpler, allowing an easy evaluation of the Lyapunov exponent.

What are the ways to attempt a further normalization of Eqs. (20) ? The first thought can be to treat them as the perturbed Galactic disk tide system with ν taken as a small parameter. Yet there are two obstacles. First, one should expect that with a quite large small parameter $\nu \approx 0.1$, the second normalization should be conducted to a relatively high order, with a quite complicated integrable kernel leading to the elliptic functions and integrals. But the second obstacle is more serious, because close to the $z = 0$ plane, the Galactic disk tides vanish, or at least become negligible when compared to the presumed, ν-dependent perturbation. Thus the complete normalization of the Galactic tides problem looks like a formidable task, that has not been achieved yet.

As the first attempt, one may try to solve the special case of Eqs. (20) for $z = 0$, when $e_3 = h_1 = h_2 = 0$. The planar case, however, becomes so simple, that it can hardly be found interesting. The level curves of the reduced Hamiltonian

$$
\mathcal{M}_1^* \;=\; -\,\nu\left(-\tfrac{5}{4}\, e_1^2 + \tfrac{5}{4}\, e_2^2 - n\,\Omega_0^{-1}\, h_3\right),
\tag{21}
$$

where $h_3 = \pm\sqrt{e_1^2 + e_2^2}$, plotted on the e_1, e_2 plane, are oval shaped curves surrounding the stable equilibrium $e = 0$. A pitchfork bifurcation occurs at the origin, but only for the semi-major axis as large as 200 kAU, i.e. at the outskirts of the Oort cloud, where the validity of the Delaunay normalization is anyway doubtful. For those who seek the approximate solution restricted to small eccentricity orbits with small inclination to the Galactic plane this is a good news, but the ones who hunt problems with rich dynamics should turn to higher inclinations.

Numerical integration of Eqs. (20) for a wide range of initial conditions has been recently performed by means of the algorithm described by Breiter et al. [3]. The results indicate the existence of numerous resonances that, in spite of the previously mentioned difficulties, can be qualitatively understood from the point of view of the ν-perturbed Galactic disk tide problem. Some examples of resonant structures are visible in Fig. 3, where the MEGNO indicator [6], closely related to the maximum Lyapunov exponent, is

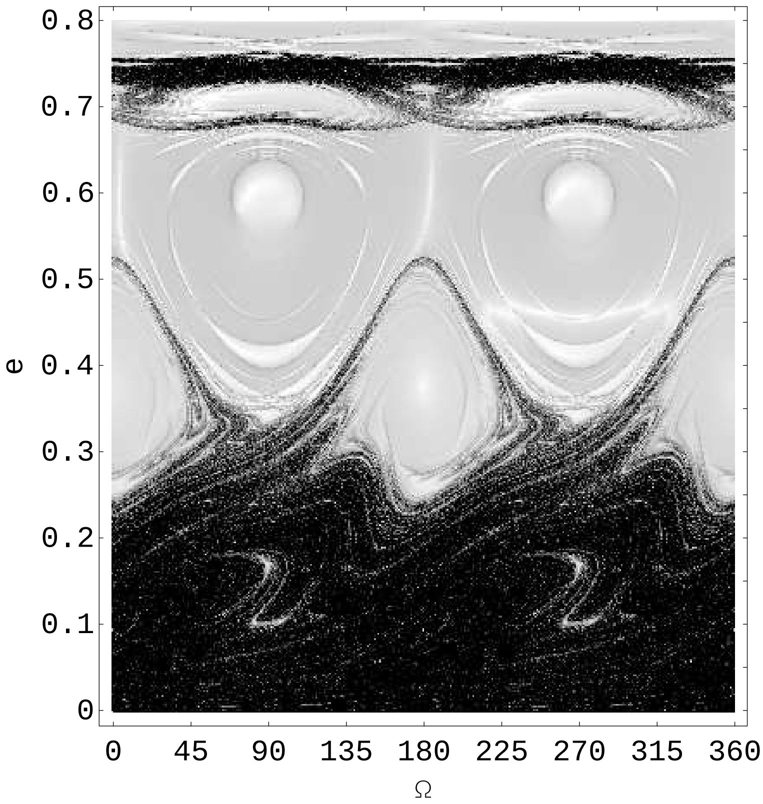

Figure 3.— MEGNO indicator for the Delaunay normalized Galactic tide. Black regions indicate a chaotic motion

plotted as a function of the initial longitude of the ascending node in the rotating frame and the initial eccentricity. The remaining initial conditions were common to all orbits: $a = 70\,\mathrm{kAU}$, $h_3 = 0.6$, $g = 90^\circ$. Black regions in Fig. 3 mark chaotic motions, whereas white areas indicate the proximity of a stable periodic orbit. It is not easy and quite risky to identify the resonances visible in the figure without a proper study of the Hamiltonian given in Eq. (18). Such studies are now being carried. Yet it becomes obvious that the longitude of the ascending node plays a significant role many of the resonance arguments. For example, the libration regions at $e \approx 0.4$ are most probably related to the critical argument $g + 2\Omega$.

4 Conclusions

The dynamics of the Oort cloud reduced to the Kepler problem with a quadratic perturbation due to the Galactic influence is an elegant problem with a wealth of unsolved questions. It deserves as much of attention as other similar problems like the Hill's case of the restricted three body problem. Its comparably low popularity may come from the fact, that it usually comes wrapped in a good dose of observational statistics. The aim of this paper is to show theoretically oriented readers what a nice dynamical system can be found behind the comets-painted screen. The author and his collaborators have been struggling with this problem for many years, yet still there is a room for many valuable contributions.

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